Solving group Steiner problems as Steiner problems: the rigorous proof

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***Abstract*—The Steiner tree problems are well-known NP-hard problems that have diverse applications. Duin et al. (2004) have intuitively proposed the widely-used transformation from the classical group Steiner tree problem to the classical Steiner tree problem in graphs. This transformation has not been rigorously proven so far. Specifically, the large M value that is used in this transformation has not been specified. In this paper, we address this issue by rigorously prove this transformation for a specific large M value.**

***Index Terms*—Graph theory, group Steiner tree problem**

I. Main content

The Steiner tree problems, which are named after Jakob Steiner, a 19th-century mathematician at the University of Berlin, have diverse applications to social, biomedical, and computer communication networks [1]. The classical Steiner Tree Problem in Graphs (STPG) [2] is about finding the minimum-cost subgraph to connect some compulsory vertices together in a connected undirected graph with positive edge costs. Many more complex Steiner tree problems in graphs have been developed based on it, including the classical Group Steiner Tree Problem (GSTP) [3], where Steiner trees must contain at least one vertex in each group of vertices. Duin et al. (2004) [4] intuitively showed that GSTP can be transformed to STPG by adding dummy vertices and edges. Even though this transformation has been widely used in the last few years (e.g. [5]), it has not been rigorously proven so far. Specifically, the large M value that is used in this transformation has not been specified. In this paper, we address this issue by rigorously prove this transformation for a specific large M value.

First, we formally define STPG and GSTP as follows.

Definition 1 (The classical Steiner Tree Problem in Graphs)**.** *Let $G(V,E,T,c)$ be a connected undirected graph, where V is the set of vertices, E is the set of edges, T is a subset of V that we refer to as compulsory vertices, and c is a function which maps each edge in E to a positive value that we refer to as edge cost. The purpose is to find a connected subgraph $G'(V',E'), T \subseteq V' \subseteq V, E' \subseteq E$ with the minimum cost $c(G') = \sum_{e \in E'} c(e)$.*

Definition 2 (The classical Group Steiner Tree Problem)**.** *Let $G(V,E,\Gamma,c)$ be a connected undirected graph, where V is the set of vertices, E is the set of edges, Γ is a collection of subsets of V that we refer to as groups, and c is a function which maps each edge in E to a positive value that we refer to as edge cost. The purpose is to find a connected subgraph $G'(V',E'), V' \subseteq V, E' \subseteq E$ with the minimum cost $c(G') = \sum_{e \in E'} c(e)$, and for each group $g \in \Gamma$, $g \cap V' \neq \emptyset$.*

We refer to the optimal solutions to GSTP and STPG as Group Steiner Minimum Tree (GSMT) and Steiner Minimum Tree (SMT) respectively. Then, we rigorously prove Duin et al.'s transformation from GSTP to STPG [4] as follows.

Theorem 1. *Let $G(V,E,\Gamma,c)$ be a connected undirected graph. For each group $g \in \Gamma$, add a compulsory vertex v_g and edges $(v_g, j)|\forall j \in g$ such that $c(v_g, j) = M = \sum_{e \in E} c(e)$. Let Θ be the GSMT on G, and Θ' be the SMT on the new graph G'. Then $\Theta = \Theta' \setminus \sum(v_g, j)$.*

Proof. Clearly, there is a feasible solution to STPG on G' whose cost is $c(\Theta) + M|\Gamma| \geq c(\Theta')$. Let Θ_1 be another tree in G' that contains all the new vertices, and every new vertex is a leaf. Suppose that there is a new vertex v_g in Θ' that is not a leaf, then

$$\begin{aligned} c(\Theta') \geq \quad & c(\Theta' \setminus \textstyle\sum(v_g, j)) + M(|\Gamma| + 1) \geq \\ c(\Theta_1) = \quad & c(\Theta_1 \setminus \textstyle\sum(v_g, j)) + M|\Gamma| \end{aligned} \tag{1}$$

which is not possible. Thus, every new vertex is a leaf in Θ', which means that 1) $c(\Theta) \geq c(\Theta') - M|\Gamma| = c(\Theta' \setminus \sum(v_g, j))$; and 2) $\Theta' \setminus \sum(v_g, j)$ is a connected tree and thus a feasible solution to GSTP on G, i.e., $c(\Theta) \leq c(\Theta' \setminus \sum(v_g, j))$. Therefore, $c(\Theta) = c(\Theta' \setminus \sum(v_g, j))$. This theorem holds. □

References

[1] Y. Sun, *Classical, prize-collecting and node-weighted Steiner tree problems in graphs*. PhD thesis, 2018.

[2] S. E. Dreyfus and R. A. Wagner, "The Steiner problem in graphs," *Networks*, vol. 1, no. 3, pp. 195–207, 1971.

[3] E. Ihler, G. Reich, and P. Widmayer, "Class Steiner trees and VLSI-design," *Discrete Applied Mathematics*, vol. 90, no. 1-3, pp. 173–194, 1999.

[4] C. Duin, A. Volgenant, and S. Voß, "Solving group Steiner problems as Steiner problems," *European Journal of Operational Research*, vol. 154, no. 1, pp. 323–329, 2004.

[5] T. Lappas, K. Liu, and E. Terzi, "Finding a team of experts in social networks," in *Proceedings of the 15th ACM SIGKDD international conference on Knowledge discovery and data mining*, pp. 467–476, ACM, 2009.